STRONG CLEANNESS OF THE 2×2 MATRIX RING OVER A GENERAL LOCAL RING

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ABSTRACT. A ring R is called strongly clean if every element of R is the sum of a unit and an idempotent that commute with each other. A recent result of Borooah, Diesl and Dorsey [3] completely characterized the commutative local rings R for which $\mathbb{M}_n(R)$ is strongly clean. For a general local ring R and $n > 1$, however, it is unknown when the matrix ring $\mathbb{M}_n(R)$ is strongly clean. Here we completely determine the local rings R for which $\mathbb{M}_2(R)$ is strongly clean.

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1. INTRODUCTION

In this paper, all rings are associative with unity and all modules are unitary. For a module M over a ring R, the R-homomorphisms of M are written on the opposite side of their arguments, and the ring of endomorphisms of M is denoted by $\text{End}(M_R)$ or $\text{End}(_R M)$. We begin by recalling a well-known notion in ring theory. An element a in a ring R is called *strongly π-regular* if both chains $aR \supseteq a^2 R \supseteq \cdots$ and $Ra \supseteq Ra^2 \supseteq \cdots$ terminate, and the ring R is called *strongly π-regular* if every element of R is strongly π-regular (or equivalently, the chain $aR \supseteq a^2 R \supseteq \cdots$ terminates for all $a \in R$, by Dischinger [10]). Thus, one-sided perfect rings are strongly π-regular. A result of Armendariz, Fisher and Snider [1] says that for a module M_R, $\varphi \in \text{End}(M_R)$ is strongly π-regular iff $M = \ker(\varphi^n) \oplus \text{Im}(\varphi^n)$ for some $n \geq 1$ (i.e., φ is a so called Fitting endomorphism). The notion of a strongly clean ring was introduced by Nicholson [14] in 1999. An element a of a ring R is called *strongly clean* if $a = e + u$ where $e^2 = e \in R$ and u is a unit of R with $eu = ue$, and the ring R is called *strongly clean* if each of its elements is strongly clean. Clearly, local rings are strongly clean. By a result of Burgess and Menal [5], every strongly π-regular ring is strongly clean. In [14], Nicholson gave a direct proof of the result that every strongly π-regular element of a ring is strongly clean, and furthermore he offered the interesting viewpoint that strongly clean elements are natural generalizations of the strongly π-regular elements by establishing the following results: for $\varphi \in \text{End}(M_R)$, φ is strongly π-regular iff there exists a decomposition $M = P \oplus Q$ such that $\varphi : P \to P$ is an isomorphism

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and $\varphi : Q \to Q$ is nilpotent; and φ is strongly clean iff there exists a decomposition $M = P \oplus Q$ such that $\varphi : P \to P$ and $1 - \varphi : Q \to Q$ are isomorphisms.

In considering whether the class of strongly clean rings is Morita invariant, Nicholson [14] raised two questions: if R is strongly clean with $e^2 = e \in R$, is eRe strongly clean? is $\mathbb{M}_n(R)$ strongly clean? In her 2002 unpublished manuscript [15], Sánchez Campos answered the first question affirmatively and gave a counter-example to the second question. In 2004, Wang and Chen [16], independently, published a counter-example to the second question. Surprisingly, the authors of the two articles came to the same counter-example $\mathbb{Z}_{(2)}$, the localization of $\mathbb{Z}$ at the prime ideal (2). This motivated the authors of [3, 7, 8] to consider the question: when is $\mathbb{M}_n(R)$ strongly clean? Observing a pattern of the 2×2 idempotent matrices over a commutative local ring, using techniques from linear algebra the authors of [7, 8] characterized the commutative local rings R for which $\mathbb{M}_2(R)$ is strongly clean. The authors of [3] had a different approach to this question. Using Nicholson's decomposition theorem, and considering different types of factorization in $R[t]$, for each n they characterized the commutative local rings R for which $\mathbb{M}_n(R)$ is strongly clean. Thus, the above question is completely settled when R is a commutative local ring.

In this paper, we continue the study of this question, focusing on the question of when $\mathbb{M}_2(R)$ is strongly clean, for noncommutative local rings R. Following P.M. Cohn [9, p.17], a ring R is called *projective-free* if every finitely generated projective R-module is free of unique rank. In Section 2, using the aforementioned decomposition theorem of Nicholson found in [14], we prove that, for a projective-free ring R, all 'non trivial' strongly clean matrices of $\mathbb{M}_n(R)$ are similar to a certain type of block diagonal matrix. For a local ring R with $n = 2$, this simply says that $A \in \mathbb{M}_2(R)$ is strongly clean iff either A is invertible or $I - A$ is invertible or A is similar to $\begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$, where $1 - t_0, t_1 \in J(R)$. This result is put to use when we establish some (easily verifiable) criteria for a 2×2 matrix ring over a local ring to be strongly clean, and, as consequences, new families of strongly clean rings are presented in Section 3. It is noticed that the same idea can be used to characterize the local rings R for which $\mathbb{M}_2(R)$ is strongly π-regular, and this discussion is recorded in Section 4.

As usual, $\mathbb{Z}$ denotes the ring of integers. The polynomial ring over a ring R in the indeterminate t is denoted by $R[t]$. For an endomorphism σ of a ring R with $\sigma(1) = 1$, let $R[[x, \sigma]]$ denote the ring of left skew power series over R. Thus, elements of $R[[x, \sigma]]$ are power series in x with coefficients in R written on the left, subject to the relation $xr = \sigma(r)x$ for all $r \in R$. The Jacobson radical and the group of units of a ring R are denoted by $J(R)$ and $U(R)$ respectively. For an integer $n > 0$, we write $\mathbb{M}_n(R)$ for the $n \times n$ matrix ring over R whose identity element we write as I_n or I, and $\mathrm{GL}_n(R)$ for the group of all invertible $n \times n$ matrices over R.

2. Strongly clean matrix rings

In this section, we obtain some necessary and sufficient conditions for a 2×2 matrix ring over a local ring to be strongly clean, which will be used to give new families of strongly clean rings in the next section.

Let F be a free R-module with a basis $\{v_1, v_2, \ldots, v_n\}$ and let $\varphi \in \text{End}(F_R)$. Then for each $1 \leq j \leq n$,

$$\varphi(v_j) = \sum_{i=1}^{n} v_i a_{ij}$$

for some $a_{ij} \in R$. Write $M_\varphi = (a_{ij}) \in \mathbb{M}_n(R)$. It is well known that the map $\text{End}(F_R) \to \mathbb{M}_n(R)$, given by $\varphi \mapsto M_\varphi$, is a ring isomorphism. Moreover, changing the basis of F_R yields conjugate matrices (via a change of basis matrix).

We need Nicholson's characterization of strongly clean matrices, which is a transition from a result of his we are recalling.

Lemma 1. [14, Theorem 3] *Let M_R be a module. The following are equivalent for $\varphi \in \text{End}(M_R)$:*

 (1) *φ is strongly clean in $\text{End}(M_R)$.*
 (2) *There is a decomposition $M = P \oplus Q$ where P and Q are φ-invariant, and $\varphi|_P$ and $(1 - \varphi)|_Q$ are isomorphisms.*

A unit a of a ring R is strongly clean because $a = 0 + a$. If $1 - a$ is a unit of R, then a is also strongly clean because $a = 1 + (a - 1)$. A strongly clean element $a \in R$ is called a *trivial strongly clean element* if a is a unit or $1 - a$ is a unit, and is called *non-trivial* otherwise.

The next lemma is a translation of Nicholson's decomposition in Lemma 1 to matrices, but this translation is a useful tool for this paper. The hypothesis here is based on following the approach of [3], and this observation is implicitly used there.

Lemma 2. *Let R be a projective-free ring. Then $A \in \mathbb{M}_n(R)$ is a non-trivial strongly clean matrix iff A is similar to $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$, where T_0 and $I - T_1$ are both invertible and neither $I - T_0$ nor T_1 is invertible.*

Proof. Make the obvious choice of basis, and write down the (block diagonal) matrix with respect to this basis. $\square$

A local ring is projective-free (see [9, Corollary 5.5, p.22]), and this is why commutative local rings are a natural place to start looking at strongly clean matrix rings, and why the approach of [3] works.

For 2×2 matrices over a local ring R, it is clear that $A \in \mathbb{M}_2(R)$ is a non-trivial strongly clean matrix iff A is similar to $\begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$, where $1 - t_0 \in J(R)$ and $t_1 \in J(R)$.

Another class of projective-free rings are the (commutative) principal ideal domains. The claim of the next example follows by Lemma 2.

Example 3. $A \in \mathbb{M}_2(\mathbb{Z})$ *is a non-trivial strongly clean matrix iff A is similar to one of the elements in* $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right\}.$

One of the primary things that makes 2×2 matrix rings over local rings easier to deal with than general matrix rings is that all matrices which are neither a unit nor I minus a unit are similar to companion matrices, as the next lemma demonstrates.

Lemma 4. *Let R be a local ring and let $A \in \mathbb{M}_2(R)$. Then either A is invertible or $I - A$ is invertible or A is similar to* $\begin{pmatrix} 0 & w_0 \\ 1 & 1 + w_1 \end{pmatrix}$ *where $w_0, w_1 \in J(R)$.*

Proof. Let $\varphi \in \text{End}(R_R^2)$, where neither φ nor $1 - \varphi$ is invertible. We show that there exists a cyclic basis $\{x, \varphi(x)\}$ of R_R^2; with respect to this basis, φ corresponds to $\begin{pmatrix} 0 & w_0 \\ 1 & 1 + w_1 \end{pmatrix}$ for some $w_0, w_1 \in J(R)$. Write $\overline{R} = R/J(R)$ and $\bar{r} = r + J(R)$ (for $r \in R$), and note that each $\phi \in \text{End}(R_R^2)$ induces an endomorphism, denoted $\overline{\phi}$, in $\text{End}(\overline{R}_{\overline{R}}^2)$. Therefore, neither $\overline{\varphi}$ nor $1 - \overline{\varphi}$ is invertible in $\text{End}(\overline{R}_{\overline{R}}^2)$, since units lift modulo the radical. Thus, as vector spaces over $\overline{R}$, $\ker(\overline{\varphi}) \neq 0$ and $\ker(1 - \overline{\varphi}) \neq 0$, and so $\overline{R}_{\overline{R}}^2 = \ker(\overline{\varphi}) \oplus \ker(1 - \overline{\varphi})$. Take $0 \neq v \in \ker(\overline{\varphi})$ and $0 \neq w \in \ker(1 - \overline{\varphi})$. Then $\{v, w\}$ is a basis for $\overline{R}_{\overline{R}}^2$. Now, lift v and w to R^2 (keeping the same names), and let $x = v + w \in R^2$. Then $\varphi(x) = \varphi(v) + \varphi(w)$, which modulo JR^2 equals w. In particular, modulo JR^2, x and $\varphi(x)$ are $v + w$ and w, which are a basis for $\overline{R}_{\overline{R}}^2$, so x and $\varphi(x)$ span R_R^2 by Nakayama's Lemma. Moreover, $\{x, \varphi(x)\}$ is a basis for R_R^2 since every local ring is stably finite. Write $\varphi^2(x) = xa + \varphi(x)b$. Reducing modulo JR^2, this equation becomes $w = (v + w)\bar{a} + w\bar{b}$. Since $\{v + w, w\}$ is linearly independent in $\overline{R}_{\overline{R}}^2$, we conclude that $\bar{a} = 0$ and $\bar{b} = 1$ (since $w = (v + w) \cdot 0 + w$). This is, $a \in J(R)$ and $b \in 1 + J(R)$. The matrix representation of φ with respect to the basis $\{x, \varphi(x)\}$ is $\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$, with $a \in J(R)$ and $b \in 1 + J(R)$, as desired. $\square$

For a monic polynomial $h(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in R[t]$, the $n \times n$

matrix $C_h = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & -a_0 \\ 1 & 0 & 0 & \cdots & 0 & -a_1 \\ 0 & 1 & 0 & \cdots & 0 & -a_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & -a_{n-2} \\ 0 & 0 & 0 & \cdots & 1 & -a_{n-1} \end{bmatrix}$ is called the *companion matrix* of

$h(t)$. A square matrix A over R is called a companion matrix if $A = C_h$ for a monic polynomial $h(t)$ over R. Here is one observation that is true for companion matrices. Lemma 5 below and its proof were introduced to the authors by the referee in order to give a conceptual proof of Lemma 6.

Lemma 5. *If $h(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0 \in R[t]$, then $C_h^n + C_h^{n-1}a_{n-1} + \cdots + C_h a_1 + I a_0 = 0$ as matrices. (That is $C_h^n + C_h^{n-1}(a_{n-1}I) + \cdots + C_h(a_1 I) + I a_0 = 0$.)*

Proof. Let $T = C_h^n + C_h^{n-1}a_{n-1} + \cdots + C_h a_1 + I a_0$. We will show that T acts as the zero endomorphism of R_R^n, and to do so it suffices to show that $Te_i = 0$ for all i, where $\{e_1, \ldots, e_n\}$ is the standard basis for R_R^n (expressed as column vectors). By construction of C_h, $Te_1 = 0$. Note that for $a \in R$, $(aI)e_i = e_i a$ (whereas this is not generally true for elements of R_R^n). Now,

$$Te_i = C_h^n e_i + C_h^{n-1}(a_{n-1}I)e_i + \cdots + C_h(a_1 I)e_i + I a_0 e_i$$
$$= C_h^n e_i + C_h^{n-1} e_i a_{n-1} + \cdots + C_h e_i a_1 + I e_i a_0.$$

At this point, note that $e_i = C_h^{i-1}e_1$, so $Te_i = C_h^{i-1}Te_1 = 0$. $\qquad\square$

Lemma 6. *Let R be a local ring, and suppose that $w_0, w_1 \in J(R)$. Write $h(t) = t^2 - (1 + w_1)t - w_0$ and consider its companion matrix*

$$C_h = \begin{pmatrix} 0 & w_0 \\ 1 & 1 + w_1 \end{pmatrix}.$$

Then, C_h is strongly clean if and only if $h(t)$ has a left root in $J(R)$ and a left root in $1 + J(R)$.

Proof. This is essentially the argument used in [11, Theorem 3.7.2]. Note that, as matrices, $C_h^2 - C_h(1 + w_1) - Iw_0 = 0$ by Lemma 5. Now, if C_h is a strongly clean element of $\mathbb{M}_2(R)$, it must be nontrivial, so it acts as a nontrivial strongly clean endomorphism φ of $_R R^2$. So, by Lemma 1, we can find a decomposition $R^2 = P \oplus Q$ where $_R P$ and $_R Q$ are φ-invariant, φ acts as an automorphism on P and $1 - \varphi$ acts as an isomorphism on Q. Since φ is nontrivial strongly clean, P and Q each has rank 1. Pick vectors v_1 and v_2 which are bases of P and Q, respectively. Both v_1 and v_2 must have at least one coordinate which is a unit. We can multiply each of v_1 and v_2 on the left by a unit to assume that v_1 and v_2 each have a coordinate which is 1. Now, for $z \in \{v_1, v_2\}$, $(z)\varphi$ is in the span of z (since P and Q are φ-invariant), so $(z)\varphi = \lambda z$ for some λ. It is easy to see that the corresponding λ for v_1 is in $1 + J(R)$ and the other is in $J(R)$. (For instance, see Lemma 2, or simply find an explicit v_1 and v_2 modulo $J(R)$, and lift appropriately to R.) Now, $0 = z\left(C_h^2 - C_h(1 + w_1) - Iw_0\right) = \lambda^2 z - \lambda z(1 + w_1) - zw_0$. Comparing the component in which z has a 1, we have $\lambda^2 - \lambda(1 + w_1) - w_0 = 0$. The two λ which we have found were in $J(R)$ and $1 + J(R)$, so we have our left roots of the polynomial h in $J(R)$ and $1 + J(R)$.

For the reverse implication, suppose that $\lambda_1 \in J(R)$ and $\lambda_2 \in 1 + J(R)$ are left roots of h. We will produce a basis of R^2 consisting of eigenvectors of φ. Consider the row vectors $v_1 = (1, \lambda_1)$ and $v_2 = (1, \lambda_2)$ which are easily seen to be a basis for $_R R^2$ (e.g. one can easily row reduce the corresponding matrix to the identity). Note that

$$(v_i)\varphi = \left(\lambda_i, w_0 + \lambda_i(1 + w_1)\right) = \lambda_i(1, \lambda_i) = \lambda_i v_i.$$

Set $P = Rv_2$ and $Q = Rv_1$. It is clear that P and Q are φ-invariant, and that $P \oplus Q = _R R^2$, and that furthermore, φ is an isomorphism on P, and $1 - \varphi$ is an isomorphism on Q. So φ is strongly clean in $\mathrm{End}(_R R^2)$ by Lemma 1. $\qquad\square$

In [11, Theorem 3.7.2], Dorsey proved that for an arbitrary ring R, if $\mathbb{M}_n(R)$ $(n \geq 1)$ is strongly clean then for each $j \in J(R)$ $t^2 - t - j$ has a root in $J(R)$.

For convenience in stating Theorem 7, let

$$W = \{f \in R[t] : f \text{ is of degree 2, monic, and } \bar{f}(0) = \bar{f}(1) = 0\},$$

where $\bar{f}$ is the image of f in $(R/J)[t]$. Note that $f \in W$ if and only if there are $w_0, w_1 \in J(R)$ for which $f(t) = t^2 - (1 + w_1)t + w_0$.

When doing the second revision of this paper, it came to our attention that, independently, Bing-jun Li [13] has also recently proved the equivalence $(1) \Leftrightarrow (4)$ of Theorem 7.

Theorem 7. *The following are equivalent for a local ring R:*

(1) $\mathbb{M}_2(R)$ *is strongly clean.*
(2) *For any $A \in \mathbb{M}_2(R)$, either A is invertible or $I - A$ is invertible or A is similar to a diagonal matrix.*
(3) *For any $w_0, w_1 \in J(R)$, $\begin{pmatrix} 0 & w_0 \\ 1 & 1 + w_1 \end{pmatrix}$ is strongly clean.*
(4) *Every $f \in W$ has a left root in $J(R)$ and a left root in $1 + J(R)$.*
(5) *Every $f \in W$ has a left root in $J(R)$.*
(6) *Every $f \in W$ has a left root in $1 + J(R)$.*
(7) *The versions of (4) or (5) or (6) with "left root" replaced by "right root".*

Proof. $(1) \Leftrightarrow (2)$. "$\Rightarrow$" is by the notice after Lemma 2, and "$\Leftarrow$" is clear, since R is local.

$(1) \Leftrightarrow (3)$. Follows immediately from Lemma 4.

$(3) \Leftrightarrow (4)$. This follows immediately from Lemma 6.

The equivalence $(5) \Leftrightarrow (6)$ follows from the fact that $f \in W$ if and only if $g(t) = f(1 - t) \in W$. Since (4) is the same as (5) plus (6), it follows that (4), (5) and (6) are equivalent.

Finally, (1) is left-right symmetric in the sense that $\mathbb{M}_2(R)$ is strongly clean if and only if $\mathbb{M}_2(R^{op})$ is strongly clean (note that R^{op} is a local ring). The "right" analogues of statements (4)-(6) are simply the corresponding "left" statements for R^{op}, which are equivalent by the equivalence of (1)-(6) for the opposite ring R^{op}, which is local. $\square$

In [3], for a commutative local ring R, the authors defined the notion of an SRC (resp., SR) factorization of a monic polynomial over R, and proved that $\mathbb{M}_n(R)$ is strongly clean iff every monic polynomial of degree n over R has an SRC factorization. As an easy corollary of Theorem 7, there is an analog of this for the 2×2 matrix ring over a local ring. The next definition extends the notion of an SRC (resp., SR) factorization from a commutative local ring to a local ring. We are deliberately not using the term SRC, because we do not know whether the definition is the appropriate generalization of SRC for general n.

Definition 8. Let R be a local ring. A monic polynomial $f(t) \in R[t]$ is said to have a $(*)$-factorization if $f(t) = g_0(t)g_1(t) = h_1(t)h_0(t)$, where $g_0(t), g_1(t)$, $h_0(t)$, $h_1(t) \in R[t]$ are monic polynomials such that $g_0(0), g_1(1), h_0(0), h_1(1) \in U(R)$. If in addition $\overline{R}[t]\bar{g_0}(t) + \overline{R}[t]\bar{g_1}(t) = \overline{R}[t]$ and $\bar{h_0}(t)\overline{R}[t] + \bar{h_1}(t)\overline{R}[t] = \overline{R}[t]$ hold, then $f(t)$ is said to have a $(**)$-factorization.

It is interesting to compare the next result with [3, Corollary 15, Proposition 17].

Corollary 9. *The following are equivalent for a local ring R:*

    (1) $\mathbb{M}_2(R)$ *is strongly clean.*
    (2) *Every companion matrix in $\mathbb{M}_2(R)$ is strongly clean.*
    (3) *Every monic quadratic polynomial over R has a $(*)$-factorization.*
    (4) *Every monic quadratic polynomial over R has a $(**)$-factorization.*

Proof. $(1) \Leftrightarrow (2)$. This holds by the equivalence of '$(1) \Leftrightarrow (4)$' of Theorem 7.

$(1) \Rightarrow (4)$. Let $f(t) = t^2 + at + b \in R[t]$. If $f(0) \in U(R)$ or $f(1) \in U(R)$, then

$$f(t) = \begin{cases} 1 \cdot f(t) = f(t) \cdot 1, & \text{if } f(1) \in U(R); \\ f(t) \cdot 1 = 1 \cdot f(t), & \text{if } f(0) \in U(R) \end{cases}$$

is a $(**)$-factorization. So assume that $f(0), f(1) \in J(R)$. Then $b \in J(R)$ and $-a = 1 + (b - f(1)) \in 1 + J(R)$. By Theorem 7, $f(t)$ has a left root $t_0 \in J(R)$ and a left root $t_1 \in 1 + J(R)$. Thus, $f(t) = (t - t_1)(t + a + t_1) = (t - t_0)(t + a + t_0)$ is clearly a $(**)$-factorization.

$(4) \Rightarrow (3)$. It is obvious.

$(3) \Rightarrow (1)$. For $w_0, w_1 \in J(R)$, $f(t) = t^2 - (1 + w_1)t - w_0$ has a $(*)$-factorization. This clearly shows that $f(t)$ has a left root in $J(R)$ and a left root in $1 + J(R)$ by (3). Hence (1) holds by Theorem 7. $\square$

3. Applications and examples

Conditions (4)-(7) of Theorem 7 are easily verifiable criteria for a 2×2 matrix ring over a local ring to be strongly clean. We use them here to give new families of strongly clean rings.

For an ideal I of a ring R, let $\overline{R} = R/I$ and write $\bar{r} = r + I$ for $r \in R$. Further, for $f(t) = \sum a_i t^i \in R[t]$, we write $\bar{f}(t) = \sum \bar{a}_i t^i \in \overline{R}[t]$.

Definition 10. [2] A local ring R (may not be commutative) with $\overline{R} := R/J(R)$ being a field is called *Henselian* if $R[t]$ satisfies Hensel's lemma: for any monic polynomial $f(t) \in R[t]$, if $\bar{f}(t) = \alpha(t)\beta(t)$ with $\alpha(t)$, $\beta(t) \in \overline{R}[t]$ monic and coprime, then there exist unique monic polynomials $g(t)$ and $h(t)$ in $R[t]$ such that $f(t) = g(t)h(t)$, $\bar{g}(t) = \alpha(t)$, and $\bar{h}(t) = \beta(t)$.

The authors of [3] proved that matrix rings of arbitrary size over a commutative Henselian ring are all strongly clean ([3, Example 22]). With Theorem 7 in hand, the same type of proof yields the analogous result for 2×2 matrices over arbitrary Henselian rings.

Proposition 11. *Let R be a Henselian ring. Then $\mathbb{M}_2(R)$ is strongly clean.*

Proof. Let $w_0, w_1 \in J(R)$ and let $f(t) = t^2 - (1 + w_1)t - w_0$. Then $\bar{f}(t) = t^2 - t = t(t-1) \in \overline{R}[t]$. By hypothesis, there exist monic polynomials $t - a, t - b \in R[t]$ such that $f(t) = (t - a)(t - b)$ and $t - \bar{a} = t$ and $t - \bar{b} = t - 1$. It follows that $a \in J(R)$ is a left root of $f(t)$. Hence $\mathbb{M}_2(R)$ is strongly clean by Theorem 7. $\square$

A Henselian ring that is not commutative can be found in [2, Example 16]. In order to give another family of strongly clean matrix rings, we need a new notion.

Following [4], a local ring R is called *bleached* if, for all $j \in J(R)$ and $u \in U(R)$, the additive abelian group endomorphisms $l_u - r_j : R \to R$ $(x \mapsto ux - xj)$ and $l_j - r_u : R \to R$ $(x \mapsto jx - xu)$ are surjective. By [4, Example 13], some examples of bleached local rings include: commutative local rings, division rings, local rings R with $J(R)$ nil, local rings R for which some power of each element of $J(R)$ is central in R, local rings R for which some power of each element of $U(R)$ is central in R, power series rings over bleached local rings, and skew power series rings $R[[x; \sigma]]$ of a bleached local ring R with σ an automorphism of R.

Definition 12. A local ring R is called *weakly bleached* if, for all $j_1, j_2 \in J(R)$, the additive abelian group endomorphisms $l_{1+j_1} - r_{j_2}$ and $l_{j_2} - r_{1+j_1}$ are surjective.

By Nicholson [14, Example 2] (also see [4, Theorem 18]), a local ring R is weakly bleached iff the 2×2 upper triangular matrix ring $\mathbb{T}_2(R)$ is strongly clean. There exist examples, however, of local rings which are not weakly bleached (e.g. [4, Example 45]). Bleached rings are clearly weakly bleached, but the converse is not true by [4, Example 38] together with [4, Theorem 30].

The next result was known in the commutative case when $\sigma = 1_R$ (see [8, Theorem 9]).

Theorem 13. *Let R be a weakly bleached local ring and let $\sigma : R \to R$ be an endomorphism with $\sigma(J(R)) \subseteq J(R)$. Then the following are equivalent for $n \geq 1$:*
   (1) $\mathbb{M}_2(R)$ *is strongly clean.*
   (2) $\mathbb{M}_2(R[[x; \sigma]])$ *is strongly clean.*
   (3) $\mathbb{M}_2(R[x; \sigma]/(x^n))$ *is strongly clean.*

Proof. $(2) \Rightarrow (3) \Rightarrow (1)$. This follows because the image of a strongly clean ring is again strongly clean.

$(1) \Rightarrow (2)$. Let $S = R[[x; \sigma]]$. Note that $J(S) = J(R) + Sx$. By Theorem 7, it suffices to show that, for any $w_0, w_1 \in J(S)$, $t^2 - (1 + w_1)t - w_0$ has a left root in $J(S)$. Write

$$w_1 = b_0 + b_1 x + \cdots,$$
$$w_0 = c_0 + c_1 x + \cdots,$$
$$t = t_0 + t_1 x + \cdots,$$

where $b_0, c_0 \in J(R)$. Then $t^2 - t(1 + w_1) - w_0 = 0 \Leftrightarrow$

$$\begin{cases} t_0^2 - t_0(1 + b_0) - c_0 & = 0 & (P_0) \\ t_k[1 - \sigma^k(t_0) + \sigma^k(b_0)] - t_0 t_k & = [t_1 \sigma(t_{k-1}) + \cdots + t_{k-1}\sigma^{k-1}(t_1)] \\ & \quad -[t_0 b_k + \cdots + t_{k-1}\sigma^{k-1}(b_1)] - c_k & (P_k) \end{cases}$$

for $k = 1, 2, \ldots$. By Theorem 7, $t^2 - (1 + b_0)t - c_0$ has a left root $t_0 \in J(R)$. Thus, $1 - \sigma^k(t_0) + \sigma^k(b_0) \in 1 + J(R)$, so (P_k) is solvable for t_k (because R is weakly bleached) for $k = 1, 2, \ldots$. Thus, $\sum_i t_i x^i \in J(S)$ is a left root of $t^2 - (1 + w_1)t - w_0$. The proof is complete. $\square$

It is unknown if the commutative Henselian rings are exactly those commutative local rings over which the matrix rings are strongly clean (see [3, Problem 23]). The next example gives a (noncommutative) local ring R that is not Henselian such that $\mathbb{M}_2(R)$ is strongly clean.

Example 14. *Let D be a division ring and σ an endomorphism of D. Then $\mathbb{M}_2(D[[x;\sigma]])$ is strongly clean by Theorem 13. If, in particular, $D = \mathbb{C}$ and σ is the complex conjugation, then $D[[x;\sigma]]$ is not Henselian by [2, Example 17].*

The next corollary follows by Proposition 11 and Theorem 13.

Corollary 15. *If R is a weakly bleached Henselian ring and σ is an endomorphism of R with $\sigma(J(R)) \subseteq J(R)$, then $\mathbb{M}_2(R[[x;\sigma]])$ and $\mathbb{M}_2\big(R[x;\sigma]/(x^n)\big)$ are strongly clean.*

4. STRONGLY π-REGULAR MATRICES

In this section, we characterize the local rings R for which $\mathbb{M}_2(R)$ is strongly π-regular. This topic is included here mainly because the techniques involved are very similar to those in previous sections.

Lemma 16. [14] *Let M_R be a module. The following are equivalent for $\varphi \in \mathrm{End}(M_R)$:*

(1) *φ is strongly π-regular in $\mathrm{End}(M_R)$.*
(2) *There is a decomposition $M = P \oplus Q$ where P and Q are φ-invariant, and $\varphi|_P$ is an isomorphism and $\varphi|_Q$ is nilpotent.*

Units and nilpotent elements of a ring are clearly strongly π-regular elements. A strongly π-regular element $a \in R$ is called a *trivial strongly π-regular element* if a is a unit or nilpotent, and is called *non-trivial* otherwise. Because of Lemma 16, the same proof of Lemma 2 works for the next lemma, which is a translation of the decomposition in Lemma 16 to matrices. The hypothesis here is based on following the approach of [3].

Lemma 17. *Let R be a projective-free ring. Then $A \in \mathbb{M}_n(R)$ is a non-trivial strongly π-regular matrix iff A is similar to $\begin{pmatrix} T_0 & 0 \\ 0 & T_1 \end{pmatrix}$, where T_0 is an invertible matrix and T_1 is a nilpotent matrix.*

Corollary 18. *Let R be a local ring. Then $A \in \mathbb{M}_2(R)$ is a non-trivial strongly π-regular matrix iff A is similar to $\begin{pmatrix} t_0 & 0 \\ 0 & t_1 \end{pmatrix}$, where $t_0 \in U(R)$ and $t_1 \in R$ is nilpotent.*

As pointed out in [3], it follows from the results of the literature that for any commutative ring R, $\mathbb{M}_n(R)$ is strongly π-regular iff so is R and that, for a commutative local ring R, $\mathbb{M}_n(R)$ is strongly π-regular iff so is R iff $J(R)$ is nil. By [3], there exists a commutative local ring R such that $\mathbb{M}_2(R)$ is strongly clean, but not strongly π-regular. Below, we characterize the local rings R for which $\mathbb{M}_2(R)$ is strongly π-regular.

Lemma 19. *Let $A \in \mathbb{M}_2(R)$ where R is a local ring. If $A \notin \mathbb{M}_2(J(R)) \cup \mathrm{GL}_2(R)$, then A is similar to $\begin{pmatrix} 0 & w \\ 1 & r \end{pmatrix}$ where $w \in J(R)$ and $r \in R$.*

Proof. Let $\varphi \in \mathrm{End}(R_R^2)$, where $\varphi \notin J(\mathrm{End}(R_R^2))$ and φ is not invertible. We show that there exists a cyclic basis $\{x, \varphi(x)\}$ of R_R^2; with respect to this basis,

φ corresponds to $\begin{pmatrix} 0 & w \\ 1 & r \end{pmatrix}$ for some $w \in J(R)$ and $r \in R$. Because $\overline{\varphi}$ is is not a unit (since units lift modulo the radical), $\ker(\overline{\varphi}) \neq 0$, but also $\mathrm{Im}(\overline{\varphi}) \neq 0$, since $\overline{\varphi} \neq 0$ (since $\varphi \notin J(\mathrm{End}(R_R^2))$). In particular, by the rank-nullity theorem, both $\ker(\overline{\varphi})$ and $\mathrm{Im}(\overline{\varphi})$ are 1-dimensional. It follows that $\mathrm{Im}(\overline{\varphi}) \cup \ker(\overline{\varphi}) \neq \overline{R}^2$ (since a vector space is never the union of two proper subspaces). Take v outside of the union, and look at $\{v, \overline{\varphi}(v)\}$. Note that $\overline{\varphi}(v) \neq 0$. And since v is not in $\mathrm{Im}(\overline{\varphi})$, $\{v, \overline{\varphi}(v)\}$ is independent. Now, lift v to x in R^2. Then, modulo JR^2, $\{x, \varphi(x)\}$ is $\{v, \overline{\varphi}(v)\}$, which is a basis for $\overline{R}_{\overline{R}}^2$. So x and $\varphi(x)$ span R_R^2 by Nakayama's Lemma. Moreover, $\{x, \varphi(x)\}$ is a basis for R_R^2 since every local ring is stably finite. Write $\varphi^2(x) = xa + \varphi(x)b$. Reducing modulo JR^2, this equation becomes $\overline{\varphi}^2(v) = v\bar{a} + \overline{\varphi}(v)\bar{b}$. Since v is not in $\mathrm{Im}(\overline{\varphi})$, we conclude that $\bar{a} = 0$. This is, $a \in J(R)$. The matrix representation of φ with respect to the basis $\{x, \varphi(x)\}$ is $\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix}$, with $a \in J(R)$, as desired. $\square$

Lemma 20. *Let R be a local ring, and suppose that $u \in U(R)$ and $w \in J(R)$. Write $h(t) = t^2 - ut - w$ and consider its companion matrix*

$$C_h = \begin{pmatrix} 0 & w \\ 1 & u \end{pmatrix}.$$

Then, C_h is strongly π-regular if and only if $h(t)$ has two left roots, one in $U(R)$ and one which is nilpotent.

Proof. The proof of Lemma 6 proves this statement, appealing to Lemma 16 instead of Lemma 1, and making the resulting obvious changes. $\square$

Theorem 21. *The following are equivalent for a local ring R:*
  (1) $\mathbb{M}_2(R)$ *is strongly π-regular.*
  (2) $\mathbb{M}_2(J(R))$ *is nil and, for any $u \in U(R)$ and $w \in J(R)$, $t^2 - ut - w$ has two left roots, one in $U(R)$ and one in $J(R)$.*
  (3) $\mathbb{M}_2(J(R))$ *is nil and, for any $u \in U(R)$ and $w \in J(R)$, $t^2 - ut - w$ has two right roots, one in $U(R)$ and one in $J(R)$.*

Proof. $(1) \Rightarrow (2)$. (1) clearly implies that $\mathbb{M}_2(J(R))$ is nil. For $u \in U(R)$ and $w \in J(R)$, let $A = \begin{pmatrix} 0 & w \\ 1 & u \end{pmatrix}$. By (1), A is strongly π-regular. Hence, by Lemma 20, $t^2 - ut - w$ has two left roots, one in $U(R)$ and one which is nilpotent. So (2) holds.

$(2) \Rightarrow (1)$. Let $A \in \mathbb{M}_2(R)$. We want to show that A is strongly π-regular. Because $\mathbb{M}_2(J(R))$ is nil, and every nilpotent element of a ring is strongly π-regular, we may assume that $A \notin \mathbb{M}_2(J(R))$ and $A \notin \mathrm{GL}_2(R)$. Thus, by Lemma 19, we may assume that $A = \begin{pmatrix} 0 & w \\ 1 & u \end{pmatrix}$ where $u \in R$ and $w \in J(R)$; moreover, we may further assume that $u \in U(R)$, for otherwise $A^2 \in \mathbb{M}_2(J(R))$, so A is nilpotent. By (2), $t^2 - ut - w = 0$ has two left roots, one in $U(R)$ and one in $J(R)$ (which is nilpotent). Thus, by Lemma 20, A is strongly π-regular.

$(1) \Leftrightarrow (3)$. Similar to the proof of $(1) \Leftrightarrow (2)$, or alternatively, appeal to the opposite ring, as in the proof of Theorem 7. $\square$

As mentioned before, for a commutative local ring R, $\mathbb{M}_2(R)$ is strongly π-regular iff $J(R)$ is nil. As a contrast of this, there exists a local ring R with $J(R)$ locally nilpotent (thus, $\mathbb{M}_2(J(R))$ is nil), but $\mathbb{M}_2(R)$ is not strongly π-regular by [6]. For a left perfect ring R, $\mathbb{M}_n(R)$ is again left perfect, so it is strongly π-regular. It is worth noting that there exists a non commutative local ring R that is not one-sided perfect such that $\mathbb{M}_n(R)$ (for each $n \geq 1$) is strongly π-regular.

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